

Transient Adaptive Discontinuous Galerkin Method with Anisotropic Meshes

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Abstract

In this paper, we present a method for performing anisotropic adaptive computations. The discontinuous Galerkin method that is used for solving transient flow problem is briefly introduced. We develop then an error indication technique that provides directional error information in order to build a non uniform metric field. We present two sample problems involving hundred of mesh refinements, both in 2D and 3D.

Keywords: Anisotropic meshes, Discontinuous Galerkin, Transient Adaptation

1 The Discontinuous Galerkin Method

Transient flow problems involving wave propagation are of great interest in computational fluid dynamics. An accurate tracking of features like moving shocks or fluid interfaces in an Eulerian fashion implies multiple mesh adaptations in order to follow complex features of the flow. The discontinuous Galerkin method (DGM) is a good candidate for solving our problems of interest. The DGM can be regarded as an extension of finite volume methods to arbitrary orders of accuracy without the need to construct complex stencils for high-order reconstruction. We seek to determine $\mathbf{u}(\Omega, t) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbf{L}^2(\Omega)^m = V(\Omega)$ as the solution of a *system of conservation laws*

$$\partial_t \mathbf{u} + \operatorname{div} \vec{\mathbf{F}}(\mathbf{u}) = \mathbf{r}. \quad (1)$$

With the aim of constructing a Galerkin form of (1), let $(\cdot, \cdot)_\Omega$ and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denote the standard $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^2(\partial\Omega)$ scalar products respectively. Multiply equation (1) by a test function $\mathbf{w} \in V(\Omega)$, integrate over Ω and use the divergence theorem to obtain the following variational

formulation

$$(\partial_t \mathbf{u}, \mathbf{w})_\Omega - (\vec{\mathbf{F}}(\mathbf{u}), \mathbf{grad} \mathbf{w})_\Omega + \langle \vec{\mathbf{F}}(\mathbf{u}) \cdot \vec{n}, \mathbf{w} \rangle_{\partial\Omega} = (\mathbf{r}, \mathbf{w})_\Omega, \quad \forall \mathbf{w} \in V(\Omega). \quad (2)$$

The physical domain Ω is first discretized into a collection of \mathcal{N}_e elements $\mathcal{T}_e = \bigcup_{e=1}^{\mathcal{N}_e} e$ called a mesh. The continuous function space $V(\Omega)$ containing the solution of (2) is then approximated on each element e of the mesh to define a finite-dimensional space $V_e(\mathcal{T}_e)$. With discontinuous finite elements, V_e is a “broken” function space that consists in the direct sum of elementary approximations \mathbf{u}_e (we use here a polynomial basis $\mathbb{P}^q(e)$ of order q):

$$V_e(\mathcal{T}_e) = \{\mathbf{u} \mid \mathbf{u} \in \mathbf{L}^2(\Omega)^m, \mathbf{u}_e \in \mathbb{P}^q(e)^m = V_e(e)\}. \quad (3)$$

Because all approximation are disconnected, we can solve the conservation laws on each element. Now, a discontinuous basis implies that the normal trace $\mathbf{F}_n = \vec{\mathbf{F}}(\mathbf{u}) \cdot \vec{n}$ is not defined on ∂e . If \mathbf{u}_e and \mathbf{u}_{e_k} are the restrictions of solution \mathbf{u} , respectively, to element e and element e_k , a *numerical flux* $\mathbf{F}_n(\mathbf{u}_e, \mathbf{u}_{e_k})$ is usually used on each portion ∂_{e_k} of ∂e shared by element e and neighboring element e_k . We have then a DG formulation

$$(\partial_t \mathbf{u}_e, \mathbf{w})_e - (\vec{\mathbf{F}}(\mathbf{u}_e), \mathbf{grad} \mathbf{w})_e + \sum_{k=1}^{n_e} \langle \mathbf{F}_n(\mathbf{u}_e, \mathbf{u}_{e_k}), \mathbf{w} \rangle_{\partial_{e_k}} = (\mathbf{r}, \mathbf{w})_e, \quad \forall \mathbf{w} \in V_e(e), \quad (4)$$

where n_e is the number of faces of element e . Several operators are possible [1, 2]. Herein, a quasi-exact Riemann solver is used to compute the numerical fluxes and a slope limiter [3] is used to produce monotonic solutions when polynomial degrees $q > 0$ are used.

2 An Error Indicator

For $q = 1$, the following quantity is used as an error indicator:

$$|e|(\nabla \epsilon)_e = \sum_{k=1}^{n_e} \int_{\partial_{e_k}} \frac{(u_e - u_{e_k}^+)}{2} \vec{n}_k ds \quad (5)$$

where u is a scalar quantity depending on \mathbf{u} , $|e|$ is the volume of element e and $(\nabla \epsilon)_e$ is a gradient due to downwind jumps $(\mathbf{u}_e - \mathbf{u}_{e_k}^+)$ of the solution that show super-convergence [4]. We use the 3 principal directions \vec{h}_k , $k = 1, 2, 3$ of the Hessian $H_{i,j}(u) = \frac{\partial^2 u}{\partial x_i \partial x_j}$ to compute directional errors $(\nabla \epsilon)_e \cdot \vec{h}_k$. In each principal direction \vec{h}_k , we compute a directional size field that takes into account directional errors.

The mesh adaptation based on the directional size field is then performed by means of some local mesh modifications [5]. In case of transient computations, the mesh-to-mesh interpolations are performed using some local projection schemes. Those are differentiated with respect to the nature of each atomistic mesh modification operator: operators including only topological modifications, operators including only geometrical modifications, operator including both topological and geometrical modifications.

3 Examples

We present the results of two compressible inviscid flow problems involving the solution of the Euler equations [6] by a DGM. The three-dimensional Euler equations have the form (1) with $\mathbf{u} = \{\rho, \rho v_x, \rho v_y, \rho v_z, E\}^t$, $\vec{\mathbf{F}}(\mathbf{u}) = \{\rho \vec{v}, \rho v_x \vec{v} + P \vec{e}_x, \rho v_y \vec{v} + P \vec{e}_y, \rho v_z \vec{v} + P \vec{e}_z, (\rho E + P) \vec{v}\}^t$ and $\mathbf{r} = \mathbf{0}$. Here, ρ is the fluid density, \vec{v} the velocity, E the internal energy, P the pressure and \vec{e}_x , \vec{e}_y and \vec{e}_z are the unit vectors in the x , y and z directions, respectively. An equation of state of the form $P = P(\rho, E)$ is also necessary to close the system. Here, we have chosen the perfect gas equation of state $P = (\gamma - 1) \rho \left[E - \frac{\|\vec{v}\|^2}{2} \right]$ with the gas constant $\gamma = 1.4$.

One of the most widely known models of blast waves is the famous Taylor-Sedov point source solution. The Sedov problem [7] involves the self-similar evolution of a cylindrical or spherical blast wave from a delta-function initial pressure perturbation in an otherwise homogeneous medium. In our example, we have made 2 Sedov explosions collide into a square domain that contains 3 rectangular obstacles (Figure 1). In practice, we initialize the code by depositing a quantity of energy $E = 1$ into a small region of radius dr . The pressure inside this volume, P_0 , is given by $P_0 = \frac{3(\gamma-1)E}{3\pi dr^2}$. Everywhere else, the density is set to $\rho = 1$ and the velocity is null initially.

We see on Figure 1 two meshes after different numbers of adaptations. The collision of the two blasts creates a very high density region in the center of the system that rapidly becomes unstable. We observe on Figure 1 the power of anisotropic mesh refinement to capture accurately complex features resulting of the various instabilities of the flow.

We have finally run a one blast Sedov problem in 3D. One picture of the results is shown on figure 1.

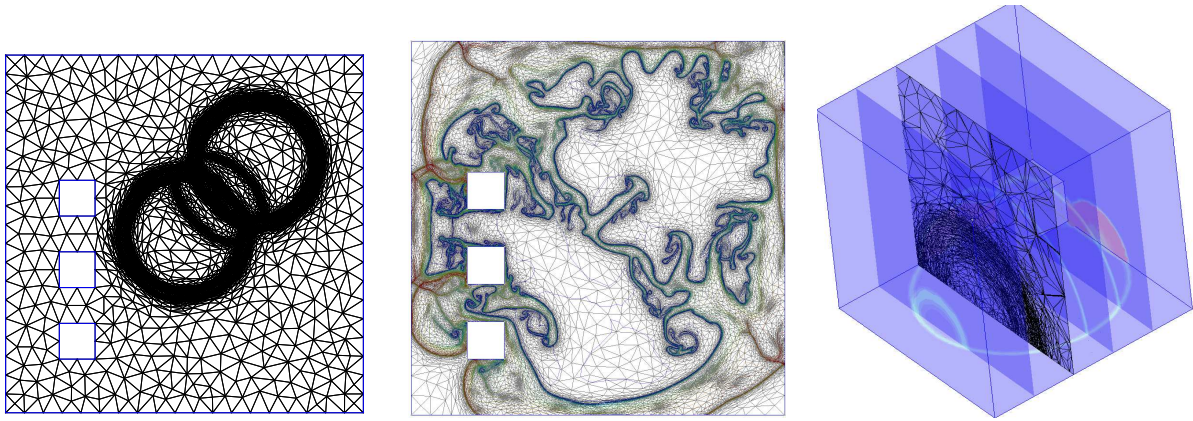


Figure 1: Mesh after 25 adaptations (left) and mesh+density contours after 227 adaptations (center) for the 2D problem. The right picture shows a cut of the mesh and some cuts of the density for the 3D Sedov problem.

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